PROPOSTAS DE RESOLUÇÃO

Capítulo 8

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Actividades de investigação

Não, porque a descoberta do tesouro não depende do ponto onde se inicia a marcha.

Localização:

- da palmeira: $P = a + bi$
- do sândalo: $S = c + di$
- do ponto de partida: $A = x + yi$

A primeira estaca, E_1, deve ser colocada no ponto de coordenadas:

$$X + (P - x) - (P - x)\mathrm{i} = a + b - y + (b - ax)\mathrm{i}$$

A segunda estaca, E_2, deve ser colocada no ponto de coordenadas:

$$X + (S - x) + (S - x)\mathrm{i} = c - d + y + (d + c - x)\mathrm{i}$$

O ponto médio entre as estacas tem de coordenadas:

$$\frac{1}{2}(E_1 + E_2) = \frac{1}{2}\left(a + b + c - d + (b - a + c + d)\mathrm{i}\right)$$

Tal como se pode verificar, as coordenadas do ponto médio entre as estacas não depende das coordenadas de A, mas somente da palmeira e do sândalo.

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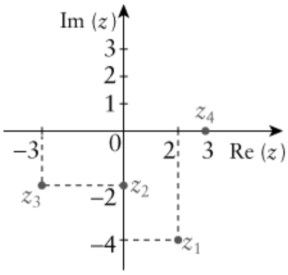
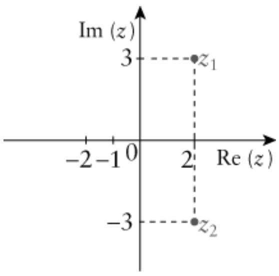
1.1

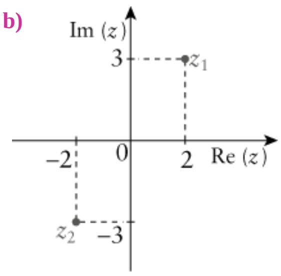


1.2 **a)**



Observação: As imagens geométricas de z_1 e z_2 são simétricas relativamente ao eixo real.

b)



Observação: As imagens geométricas de z_1 e z_2 são simétricas relativamente à origem do referencial.

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2.1 **a)** $(2 - 3\mathrm{i}) + (4 - \mathrm{i}) =$

$= 2 - 3\mathrm{i} + 4 - \mathrm{i} = 6 - 4\mathrm{i}$

b) $(-1 + \mathrm{i}) - (2 - 3\mathrm{i}) = -1 + \mathrm{i} - 2 + 3\mathrm{i} =$

$= -3 + 4\mathrm{i}$

c) $(2 + \mathrm{i}) - 3\mathrm{i} = 2 + \mathrm{i} - 3\mathrm{i} = 2 - 2\mathrm{i}$

d) $\left(\frac{3}{2} + \mathrm{i}\right) - \left(\frac{3}{4} - \mathrm{i}\right) = \frac{3}{2} + \mathrm{i} - \frac{3}{4} + \mathrm{i} =$

$= \frac{6}{4} - \frac{3}{4} + 2\mathrm{i} = \frac{3}{4} + 2\mathrm{i}$

e) $\left(\frac{1}{2} - 2\mathrm{i}\right) + \left(-1 + \frac{1}{2}\mathrm{i}\right) = \frac{1}{2} - 2\mathrm{i} - 1 + \frac{1}{2}\mathrm{i} =$

$= \frac{1}{2} - \frac{2}{2} - \frac{4}{2}\mathrm{i} + \frac{1}{2}\mathrm{i} = -\frac{1}{2} - \frac{3}{2}\mathrm{i}$

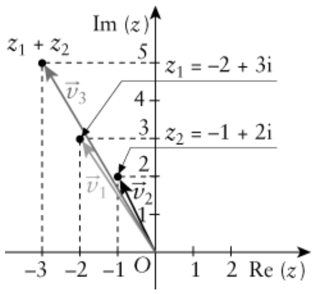
2.2 $z_1 + z_2 = (a + bi) + (c + di) = a + bi + c + di$

$= (a + c) + (b + d)\,\mathrm{i}$

2.3 **a)** $z_1 + z_2 = (-2 + 3\mathrm{i}) + (-1 + 2\mathrm{i})$

$= -2 + 3\mathrm{i} - 1 + 2\mathrm{i}$

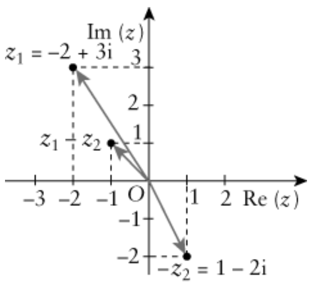
$= -3 + 5\mathrm{i}$



b) $z_1 - z_2 = (-2 + 3\mathrm{i}) - (-1 + 2\mathrm{i})$

$= -2 + 3\mathrm{i} + 1 - 2\mathrm{i}$

$= -1 + \mathrm{i}$



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3.1 **a)** $(2 + i)i = 2i + i^2$

$= 2i + (-1) = 2i - 1 = -1 + 2i$

b) $i(1 - 5i) = i - 5i^2 = i - 5(-1) = i + 5 = 5 + i$

c) $(2-i)\left(3+\frac{5}{2}i\right) = 6+\frac{10}{2}i - 3i - \frac{5}{2}i^2$

$= 6 + 5i - 3i - \frac{5}{2}(-1) = 6 + 2i + \frac{5}{2}$

$= \frac{12}{2} + \frac{5}{2} + 2i = \frac{17}{2} + 2i$

d) $(-3+i)\left(-\frac{1}{2}-i\right) = \frac{3}{2} + 3i - \frac{1}{2}i - i^2$

$= \frac{3}{2} + \frac{6}{2}i - \frac{1}{2}i - (-1) = \frac{3}{2} + \frac{5}{2}i + 1$

$= \frac{5}{2} + \frac{5}{2}i$

3.2 $(a + bi)(c + di) = ac + adi + bci + bdi^2$

$= ac + adi + bci - bd = (ac - bd) + (ad + bc)i$

4.1 $(2 - 3i)(2 + 3i) = 2^2 - (3i)^2 = 4 - 9i^2 = 4 + 9 = 13$

4.2 $(1 - i)(1 + i) = 1^2 - i^2 = 1 + 1 = 2$

4.3 $(-1 + i)(i + 1) = (i - 1)(i + 1) = i^2 - 1^2$

$= -1 - 1 = -2$

5.1 $(-2 + 3i)^2 = 4 - 12i + 9^2 = 4 - 12i - 9 = -5 - 12i$

5.2 $\left(-1-\frac{1}{2}i\right)^2 = 1 + i + \frac{1}{4}i^2 = 1 + i - \frac{1}{4} = \frac{3}{4} + i$

5.3 $(-1 + i)^3 = (-1 + i)^2(-1 + i)$

$= (1 - 2i + i^2)(-1 + i) = (1 - 2i - 1)(-1 + i)$

$= (-2i)(-1 + i) = 2i - 2i^2 = 2 + 2i$

5.4 $(a + bi)^2 = a^2 + 2ab\,i + (bi)^2 = a^2 + 2ab\,i + b^2 i^2$

$= a^2 + 2ab\,i - b^2 = (a^2 - b^2) + 2ab\,i$

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6.1 $\frac{1-2i}{i} = \frac{(1-2i)(-i)}{i\times(-i)} = \frac{-i+2i^2}{-i^2}$

$= \frac{-1-2}{1} = -2 - i$

6.2 $\frac{1-3i}{1+i} = \frac{(1-3i)(1-i)}{(1+i)(1-i)} = \frac{1-i-3i+3i^2}{1^2-i^2}$

$= \frac{1-4i-3}{1+1} = \frac{-2-4i}{2} = -1 - 2i$

6.3 $\frac{3}{-1+2i} = \frac{3(-1-2i)}{(-1+2i)(-1-2i)}$

$= \frac{-3-6i}{(-1)^2-(2i)^2} = \frac{-3-6i}{1-4i^2}$

$= \frac{-3-6i}{5} = -\frac{3}{5} - \frac{6}{5}i$

6.4 $(2-3i)^{-1} = \frac{1}{2-3i} = \frac{1\times(2+3i)}{(2-3i)(2+3i)}$

$= \frac{2+3i}{4-9i^2} = \frac{2+3i}{13} = \frac{2}{13} + \frac{3}{13}i$

6.5 $\frac{2+3i}{z} = 1 - 5i \Leftrightarrow \frac{2+3i}{1-5i} = z$

$\Leftrightarrow \frac{(2+3i)(1+5i)}{(1-5i)(1+5i)} = z$

$\Leftrightarrow \frac{2+10i+3i+15i^2}{1-25i^2} = z$

$\Leftrightarrow \frac{2+13i-15}{1+25} = z \Leftrightarrow \frac{-13+13i}{26} = z$

$\Leftrightarrow -\frac{1}{2} + \frac{1}{2}i = z$

6.6 $\frac{a+bi}{c+di} = \frac{(a+bi)(c-di)}{(c+di)(c-di)} = \frac{ac-adi+bci-bdi^2}{c^2-d^2i^2}$

$= \frac{ac-adi+bci+bd}{c^2+d^2} = \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}i$

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7.1 $i^{39} = (i^4)^9 \times i^3 = i^3 = -i$

Cálculo auxiliar

39	4
03	9

7.2 $i^{37} + i^{999} - 2i^{25} =$

$= (i^4)^9 \times i + (i^4)^{249} \times i^3 - 2(i^4)^6 i$

$= i + i^3 - 2i$

$= i - i - 2i$

$= -2i$

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8. Consideremos que $a + bi$ é uma raiz quadrada de $-3 + 4\text{i}$.

Então:

$$-3+4\text{i}=(a+b\text{i})^2 \Leftrightarrow -3+4\text{i}=a^2+2ab\text{i}-b^2$$

$$\Leftrightarrow \begin{cases} -3=a^2-b^2 \\ 2ab=4 \end{cases} \Leftrightarrow \begin{cases} -3=a^2-\left(\dfrac{2}{a}\right)^2 \\ b=\dfrac{2}{a}, a\neq 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} -3=a^2-\dfrac{4}{a^2} \qquad (1) \\ b=\dfrac{2}{a} \end{cases}$$

Resolvendo a equação (1), vem que:

$$-3=a^2-\frac{4}{a^2} \Leftrightarrow -3a^2=a^4-4$$

$$\Leftrightarrow a^4+3a^2-4=0$$

Substituindo $a^2 = y$, vem:

$$y^2+3y-4=0 \Leftrightarrow y=\frac{-3\pm\sqrt{9+16}}{2}$$

$$\Leftrightarrow y=\frac{-3\pm 5}{2}$$

$$\Leftrightarrow y=-4 \vee y=1$$

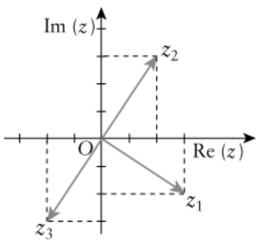
$y=-4$ é impossível, uma vez que $y = a^2$ e $a\in\mathbb{R}$

$y=1 \Leftrightarrow a^2=1 \Leftrightarrow a=-1 \vee a=1$

- Se $a = -1$ então $b = -2$.
- Se $a = 1$ então $b = 2$.

Logo, as raízes quadradas de $-3 + 4\text{i}$ são $-1 - 2\text{i}$ e $1 + 2\text{i}$.

9.



$z_1 = 3 - 2\text{i}$

$z_2 = \text{i}\,(3 - 2\text{i}) = 3\text{i} - 2\text{i}^2 = 2 + 3\text{i}$

$z_3 = -\text{i}\,(3 - 2\text{i}) = -3\text{i} + 2\text{i}^2 = -2 - 3\text{i}$

9.1 $z_1 \curvearrowright z_2 \rightarrow R\,(0, 90°)$

9.2 $z_1 \curvearrowright z_3 \rightarrow R\,(0, -90°)$

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10. Mostrar que $\overline{z_1+z_2}=\overline{z_1}+\overline{z_2}$

Pretende-se mostrar que o conjugado da soma de dois complexos é igual à soma dos conjugados das parcelas.

Sejam $z_1 = a + bi$ e $z_2 = c + di$

1.º membro $= \overline{z_1+z_2}$

$$=\overline{(a+b\text{i})+(c+d\text{i})}$$

$$=\overline{(a+c)+(b+d)\text{i}}$$

$$=(a+c)-(b+d)\text{i}$$

2.º membro $= \overline{z_1}+\overline{z_2}$

$$=\overline{(a+b\text{i})}+\overline{(c+d\text{i})}$$

$$=(a-b\text{i})+(c-d\text{i})$$

$$=(a+c)-(b+d)\text{i}$$

= 1.º membro

Logo, $\overline{z_1+z_2}=\overline{z_1}+\overline{z_2}$ c.q.m.

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11.1 $z^2+4=0 \Leftrightarrow z^2=-4$

$$\Leftrightarrow z=\pm\sqrt{-4} \Leftrightarrow z=\pm\text{i}\sqrt{4}$$

$$\Leftrightarrow z=-2\text{i} \vee z=2\text{i}$$

11.2 $z^2+10z+26=0$

$$\Leftrightarrow z=\frac{-10\pm\sqrt{100-4\times 26}}{2}$$

$$\Leftrightarrow z=\frac{-10\pm\sqrt{-4}}{2}$$

$$\Leftrightarrow z=\frac{-10\pm 2\text{i}}{2}$$

$$\Leftrightarrow z=-5-\text{i} \vee z=-5+\text{i}$$

11.3 $16z^2+8z+5=0$

$$\Leftrightarrow z=\frac{-8\pm\sqrt{64-4\times16\times5}}{2\times16}$$

$$\Leftrightarrow z=\frac{-8\pm\sqrt{-256}}{32} \Leftrightarrow z=\frac{-8\pm16\text{i}}{32}$$

$$\Leftrightarrow z=-\frac{1}{4}-\frac{1}{2}\text{i} \vee z=-\frac{1}{4}+\frac{1}{2}\text{i}$$

11.4 $z^3+16z=0 \Leftrightarrow z\left(z^2+16\right)=0$

$\Leftrightarrow z=0 \vee z^2=-16$

$\Leftrightarrow z=0 \vee z=\pm\sqrt{-16}$

$\Leftrightarrow z=0 \vee z=-4i \vee z=4i$

11.5 $z^3+6z^2+13z=0 \Leftrightarrow z\left(z^2+6z+13\right)=0$

$\Leftrightarrow z=0 \vee z^2+6z+13=0$

$\Leftrightarrow z=0 \vee z=\dfrac{-6\pm\sqrt{36-4\times 13}}{2}$

$\Leftrightarrow z=0 \vee z=\dfrac{-6\pm\sqrt{-16}}{2}$

$\Leftrightarrow z=0 \vee z=\dfrac{-6\pm 4i}{2}$

$\Leftrightarrow z=0 \vee z=-3-2i \vee z=-3+2i$

11.6 $z^2-2iz-5=0 \Leftrightarrow z=\dfrac{2i\pm\sqrt{(2i)^2-4\times(-5)}}{2}$

$\Leftrightarrow z=\dfrac{2i\pm\sqrt{-4+20}}{2}$

$\Leftrightarrow z=\dfrac{2i\pm 4}{2}$

$\Leftrightarrow z=-2+i \vee z=2+i$

11.7 $z^3+6iz^2=10z \Leftrightarrow$

$\Leftrightarrow z^3+6iz^2-10z=0$

$\Leftrightarrow z\left(z^2+6iz-10\right)=0$

$\Leftrightarrow z=0 \vee z^2+6iz-10=0$

$\Leftrightarrow z=0 \vee z=\dfrac{-6i\pm\sqrt{(6i)^2+40}}{2}$

$\Leftrightarrow z=0 \vee z=\dfrac{-6i\pm\sqrt{-36+40}}{2}$

$\Leftrightarrow z=0 \vee z=\dfrac{-6i\pm 2}{2}$

$\Leftrightarrow z=0 \vee z=1-3i \vee z=-1-3i$

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12.1 $z_1=3$

módulo: $|z_1|=3$

argumento: $\arg z_1=0$

12.2 $z_2 = 2 + 2i$

módulo: $|z_2|=\sqrt{2^2+2^2}=\sqrt{8}=2\sqrt{2}$

$\begin{cases} \operatorname{tg}\left(\arg z_1\right)=\dfrac{2}{2}=1 \\ \arg z_1 \in 1.^{\text{o}}\,Q \end{cases} \Rightarrow \dfrac{\pi}{4}$ é argumento de z_1

12.3 $z_3 = 5i$

módulo: $|z_3|=5$

argumento: $\arg z_3=\dfrac{\pi}{2}$

12.4 $z_4=-3+\sqrt{3}i$, a imagem geométrica de

$z_4 \curvearrowright \left(-3,\sqrt{3}\right)\in 2.^{\text{o}}\,Q$

módulo: $|z_4|=\sqrt{(-3)^2+\left(\sqrt{3}^2\right)}=\sqrt{9+3}$

$=\sqrt{12}=2\sqrt{3}$

$\begin{cases} \operatorname{tg}\left(\arg z_4\right)=\dfrac{\sqrt{3}}{-3} \\ \arg z_4 \in 2.^{\text{o}}\,Q \end{cases} \Rightarrow \pi-\dfrac{\pi}{6}=\dfrac{5\pi}{6}$ é argumento de z_4

12.5 $z_5=-2\in\mathbb{R}^-$

módulo: $|z_5|=2$

argumento: $\arg z_5=\pi$

12.6 $z_6=-\sqrt{2}i$

módulo: $|z_6|=\left|-\sqrt{2}i\right|\sqrt{2}$

argumento: $\arg z_6=\dfrac{3\pi}{2}$

12.7 $z_7=1-\sqrt{3}i$

módulo: $|z_7|=\sqrt{1^2+\left(-\sqrt{3}\right)^2}=\sqrt{1+3}=2$

$\begin{cases} \operatorname{tg}\left(\arg z_7\right)=\dfrac{-\sqrt{3}}{1}=-\sqrt{3} \\ \arg z_7 \in 4.^{\text{o}}\,Q \end{cases} \Rightarrow -\dfrac{\pi}{3}$ é argumento de z_7

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13.1 $z_1=3\in\mathbb{R}^+$

Logo, $\arg z_1=0$ e $|z_1|=3$,

daí que $z_1 = 3 \operatorname{cis} 0$.

13.2 $z_2 = \sqrt{3} + i$

A imagem geométrica de

$z_2 \curvearrowright (\sqrt{3}, 1) \in 1.^{o} Q$

$$\begin{cases} \operatorname{tg}(\arg z_2) = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} \Rightarrow \frac{\pi}{6} \text{ é argumento de } z_1 \\ \arg z_1 \in 1.^{o} Q \end{cases}$$

$$|z_2| = \sqrt{(\sqrt{3})^2 + 1^2} = 2$$

daí que $z_2 = 2\operatorname{cis}\left(\frac{\pi}{6}\right)$.

13.3 $z_3 = 4i$

$\arg z_3 = \frac{\pi}{2}$ e $|z_3| = 4$

daí que $z_3 = 4\operatorname{cis}\left(\frac{\pi}{2}\right)$.

13.4 $z_4 = -4 - 4i$

A imagem geométrica de

$z_4 \curvearrowright (-4, -4) \in 3.^{o} Q$

$$\begin{cases} \operatorname{tg}(\arg z_4) = \frac{-4}{-4} = 1 \\ \arg z_4 \in 3.^{o} Q \end{cases} \Rightarrow \pi + \frac{\pi}{4} = \frac{5\pi}{4} \text{ é argumento de } z_4$$

$$|z_4| = \sqrt{|-4|^2 + (-4)^2} = \sqrt{16+16} = \sqrt{32} = 4\sqrt{2}$$

daí que $z_4 = 4\sqrt{2}\operatorname{cis}\frac{5\pi}{4}$.

13.5 $z_5 = -2i$

$\arg z_5 = \frac{3\pi}{2}$ e $|z_5| = 2$,

daí que $z_5 = 2\operatorname{cis}\frac{3\pi}{2}$.

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14.1 $z = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

$$= 2\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = \frac{2}{2} + \frac{2\sqrt{3}}{2}i = 1 + \sqrt{3}i$$

14.2 $5\operatorname{cis}\left(-\frac{\pi}{4}\right) =$

$$= 5\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right) = 5\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right)$$

$$= \frac{5\sqrt{2}}{2} - \frac{5\sqrt{2}}{2}i$$

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15.1 **b)** $z = \rho\operatorname{cis}\theta$

$= \rho\cos\theta + \rho i\sin\theta$

$\bar{z} = \rho\cos\theta - \rho i\sin\theta$

$= \rho(\cos\theta - i\sin\theta)$

$= \rho(\cos(-\theta) + i\sin(-\theta))$

$= \rho\operatorname{cis}(-\theta)$

b) $z \times \bar{z} = |z|^2$

Seja $z = \rho\operatorname{cis}\theta$ então $\bar{z} = \rho\operatorname{cis}(-\theta)$

Donde $z \times \bar{z} = (\rho\operatorname{cis}\theta) \times (\rho\operatorname{cis}(-\theta))$

$= \rho^2\operatorname{cis}[\theta + (-\theta)]$

$= \rho^2\operatorname{cis}0$

$= \rho^2[\cos 0 + i\sin 0]$

$= \rho^2[1 + i \times 0]$

$= \rho^2 = |z|^2$

Logo, $z \times \bar{z} = |z|^2$.

c) $\frac{1}{z} = \frac{\bar{z}}{|z|^2}$

Seja $z = \rho\operatorname{cis}\theta$, então $\bar{z} = \rho\operatorname{cis}(-\theta)$ e $|z| = \rho$

Temos que:

1.º membro $= \frac{1}{z} = \frac{1}{\rho\operatorname{cis}(\theta)} = \frac{\operatorname{cis}(\theta)}{\rho\operatorname{cis}(\theta)\operatorname{cis}(-\theta)}$

$= \frac{\operatorname{cis}(\theta)}{\rho\operatorname{cis}0} = \frac{\operatorname{cis}(-\theta)}{\rho}$

2.º membro $= \frac{\bar{z}}{|z|^2} = \frac{\rho\operatorname{cis}(-\theta)}{\rho^2}$

$= \frac{\operatorname{cis}(-\theta)}{\rho}$ = 1.º membro

De outro modo:

Seja: $z = a + bi$,então: $\overline{z} = a - bi$ e $|z| = \sqrt{a^2 + b^2}$

1.º membro = $\frac{1}{z} = \frac{1}{a + bi}$

$= \frac{a + bi}{(a + bi)(a - bi)} = \frac{a - bi}{a^2 - b^2 i^2}$

$= \frac{a - bi}{a^2 + b^2}$

2.º membro = $\frac{\overline{z}}{|z|^2} = \frac{a - bi}{\left(\sqrt{a^2 + b^2}\right)^2} = \frac{a - bi}{a^2 + b^2}$

= 1.º membro

15.2 **a)** $z_1 \times z_2 = \left[\frac{1}{3}\text{cis}\left(\frac{\pi}{4}\right)\right] \times \left[5\text{cis}\left(-\frac{\pi}{4}\right)\right]$

$= \frac{1}{3} \times 5\text{cis}\left[\frac{\pi}{4} + \left(-\frac{\pi}{4}\right)\right] = \frac{5}{3}\text{cis}0$

b) $z_1 \times z_3 = \left[\frac{1}{3}\text{cis}\left(\frac{\pi}{4}\right)\right] \times \left[3\text{cis}\left(\frac{3\pi}{4}\right)\right]$

$= \frac{1}{3} \times 3\text{cis}\left(\frac{\pi}{4} + \frac{3\pi}{4}\right)$

$= 1\text{cis}\left(\frac{4\pi}{4}\right) = \text{cis}\,\pi$

c) $z_1 \times z_2 \times z_3 =$

$= (z_1 \times z_2) \times z_3$ pela propriedade associativa da multiplicação

$= \left[\frac{5}{3}\text{cis}0\right] \times \left[3\text{cis}\frac{3\pi}{4}\right]$ pela alínea **a)**

$= \frac{5}{3} \times 3\text{cis}\left(0 + \frac{3\pi}{4}\right) = 5\text{cis}\left(\frac{3\pi}{4}\right)$

15.3 Há uma infinidade de soluções.

Por exemplo: $2\,\text{cis}\,\pi \times 2\,\text{cis}(-\pi) = 4\,\text{cis}\,0$

$4\,\text{cis}\frac{\pi}{2} \times \text{cis}\left(-\frac{\pi}{2}\right) = 4\,\text{cis}\,0$

15.4 **a)** $\overline{z_1} \times z_1 = \left(\sqrt{6} + \sqrt{2}i\right) \times \left(\sqrt{6} - \sqrt{2}i\right)$

$= \left[2\sqrt{2}\text{cis}\left(\frac{\pi}{6}\right)\right] \times \left[2\sqrt{2}\text{cis}\left(-\frac{\pi}{6}\right)\right] \frac{5}{3}$

$= 8\text{cis}\left[\frac{\pi}{6} + \left(-\frac{\pi}{6}\right)\right] = 8\text{cis}0 = 8$

Cálculo auxiliar

Seja $z_1 = \sqrt{6} - \sqrt{2}i$, então:

$|z_1| = \sqrt{\left(\sqrt{6}\right)^2 + \left(-\sqrt{2}\right)^2} = \sqrt{6 + 2} = 2\sqrt{2}$

Assim:

$\begin{cases} \text{tg}(\arg z_1) = \frac{-\sqrt{2}}{\sqrt{6}} = -\frac{\sqrt{3}}{3} \Rightarrow \\ \arg z_1 \in 4.^{\text{o}} Q \end{cases}$

$\Rightarrow -\frac{\pi}{6}$ é um argumento de z_1

Donde:

$z_1 = 2\sqrt{2}\,\text{cis}\left(-\frac{\pi}{6}\right)$ e $\overline{z_1} = 2\sqrt{2}\,\text{cis}\left(\frac{\pi}{6}\right)$

b) $z_1 \times z_2 \times \overline{z_3}$

$= 2\sqrt{2}\text{cis}\left(-\frac{\pi}{6}\right) \times 2\text{cis}\left(\frac{3\pi}{2}\right) \times 2\text{cis}(-\pi)$

$= \left(2\sqrt{2} \times 2 \times 2\right)\text{cis}\left(-\frac{\pi}{6} + \frac{3\pi}{2} - \pi\right)$

$= 8\sqrt{2}\text{cis}\left(-\frac{\pi}{6} + \frac{9\pi}{6} - \frac{6\pi}{6}\right)$

$= 8\sqrt{2}\text{cis}\left(\frac{\pi}{3}\right)$

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16.1 **a)** z^{29} , sendo $z = 1\text{cis}\,\pi$

$z^{29} = (1\text{cis}\,\pi)^{29} = 1^{29}\,\text{cis}(29\pi) =$

$= 1\text{cis}(\pi) = \text{cis}\pi$

b) $\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)^{80}$

$= \left[1\text{cis}\left(\frac{7\pi}{6}\right)\right]^{80}$

$= 1^{80}\,\text{cis}\left(\frac{7\pi}{6} \times 80\right) = \text{cis}\left(\frac{560\pi}{6}\right) =$

$= \text{cis}\left(\frac{4\pi}{3}\right) = \text{cis}\left(-\frac{2\pi}{3}\right)$

Cálculo auxiliar

Seja $z_1 = -\frac{\sqrt{3}}{2} - \frac{1}{2}i$, então:

$|z_1| = \sqrt{\left(-\frac{\sqrt{3}}{2}\right)^2 + \left(-\frac{1}{2}\right)^2} = \sqrt{\frac{3}{4} + \frac{1}{4}} = 1$

$\text{tg}(\arg z_1) = \frac{\sqrt{3}}{3} \Rightarrow$ $(\arg(z_1) \in 3.^{\text{o}} Q)$

$= \pi + \frac{\pi}{6} = \frac{7\pi}{6}$ é um argumento de z_1

$z_1 = \text{cis}\frac{7\pi}{6}$

c) $(1-\mathrm{i})^{-100} = \left[\sqrt{2}\,\mathrm{cis}\left(-\frac{\pi}{4}\right)\right]^{-100}$

$= \left(\sqrt{2}\right)^{-100} \mathrm{cis}\left(-\frac{\pi}{4}\times(-100)\right)$

$= \frac{1}{\left(\sqrt{2}\right)^{100}} \mathrm{cis}(25\pi) = \frac{1}{\left(2^{\frac{1}{2}}\right)^{100}} \mathrm{cis}\,\pi$

$= \frac{1}{2^{50}} \mathrm{cis}\,\pi$

16.2 Seja $z = \mathrm{cis}\theta$

a) $z + \bar{z} = 2\cos\theta$

Se $z = \mathrm{cis}\theta$ então $\bar{z} = \mathrm{cis}(-\theta)$, sendo:

1.º membro = $z + \bar{z} = \mathrm{cis}\theta + \mathrm{cis}(-\theta)$

$= (\cos\theta + \mathrm{i}\sin\theta) + (\cos(-\theta) + \mathrm{i}\sin(-\theta))$

$= \cos\theta + \mathrm{i}\sin\theta + \cos\theta - \mathrm{i}\sin\theta$

$= 2\cos\theta$ = 2.º membro

b) $z^2 + \bar{z}^2 = 2\cos(2\theta)$

1.º membro = $z^2 + \bar{z}^2 = (\mathrm{cis}\theta)^2 + (\mathrm{cis}(-\theta))^2$

$= \mathrm{cis}(2\theta) + \mathrm{cis}(-2\theta)$

$= \cos(2\theta) + \mathrm{i}\sin(2\theta) +$

$+ \cos(-2\theta) + \mathrm{i}\sin(-2\theta)$

$= \cos(2\theta) + \mathrm{i}\sin(2\theta) +$

$+ \cos(2\theta) - \mathrm{i}\sin(2\theta)$

$= 2\cos(2\theta)$ = 2.º membro

c) $z^n + \bar{z}^n = 2\cos(n\theta)$

1.º membro = $z^n + \bar{z}^n = (\mathrm{cis}\theta)^n + (\mathrm{cis}(-\theta))^n$

$= \mathrm{cis}(\mathrm{n}\theta) + \mathrm{cis}(-n\theta)$

$= \cos(\mathrm{n}\theta) + \mathrm{i}\sin(n\theta) +$

$+ \cos(-n\theta) + \mathrm{i}\sin\theta(-n\theta)$

$= \cos(n\theta) + \mathrm{i}\sin(n\theta) +$

$+ \cos(\mathrm{n}\theta) - \mathrm{i}\sin(\mathrm{n}\theta)$

$= 2\cos(n\theta)$ = 2.º membro

d) $\sin(n\theta) = \frac{z^n - \bar{z}^n}{2\mathrm{i}}$

2.º membro = $\frac{z^n - \bar{z}^n}{2\mathrm{i}}$

$= \frac{(\mathrm{cis}2\theta)^n - \left(\mathrm{cis}(-\theta)^n\right)}{2\mathrm{i}}$

$= \frac{\mathrm{cis}(n\theta) - \mathrm{cis}(-n\theta)}{2\mathrm{i}}$

$= \frac{\cos(n\theta) + \mathrm{i}\sin(n\theta) - (\cos(-n\theta) + \mathrm{i}\sin(-n\theta))}{2\mathrm{i}}$

$= \frac{\cos(n\theta) + \mathrm{i}\sin(n\theta) + \cos(n\theta) + \mathrm{i}\sin(n\theta)}{2\mathrm{i}}$

$= \frac{2\mathrm{i}\sin(n\theta)}{2\mathrm{i}}$

$= \sin(n\theta)$ = 1.º membro

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17.1 $\frac{2\mathrm{i}}{\mathrm{cis}\left(\frac{\pi}{6}\right)} = \frac{2\mathrm{cis}\frac{\pi}{2}}{\mathrm{cis}\left(\frac{\pi}{6}\right)}$

$= 2\mathrm{cis}\left(\frac{\pi}{2} - \frac{\pi}{6}\right) = 2\mathrm{cis}\left(\frac{3\pi}{6} - \frac{\pi}{6}\right) = 2\mathrm{cis}\left(\frac{\pi}{3}\right)$

17.2 $\left(\frac{\frac{\sqrt{3}}{2} + \frac{1}{2}\mathrm{i}}{1+\mathrm{i}}\right)^2 = \left[\frac{\mathrm{cis}\left(\frac{\pi}{6}\right)}{\sqrt{2}\mathrm{cis}\left(\frac{\pi}{4}\right)}\right]^2$

$= \left[\frac{1}{\sqrt{2}}\mathrm{cis}\left(\frac{\pi}{6} - \frac{\pi}{4}\right)\right]^2$

$= \left[\frac{\sqrt{2}}{2}\mathrm{cis}\left(\frac{2\pi}{12} - \frac{3\pi}{12}\right)\right]^2$

$= \left[\frac{\sqrt{2}}{2}\mathrm{cis}\left(-\frac{\pi}{12}\right)\right]^2$

$= \left(\frac{\sqrt{2}}{2}\right)^2 \mathrm{cis}\left(-\frac{\pi}{12}\times 2\right) = \frac{1}{2}\mathrm{cis}\left(-\frac{\pi}{6}\right)$

Cálculo auxiliar

Seja $z_1 = \frac{\sqrt{3}}{2} + \frac{1}{2}\mathrm{i}$, então:

$|z_1| = \sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{3}{4} + \frac{1}{4}} = 1$

$\text{tg}(\arg z_1) = \frac{\sqrt{3}}{3} \Rightarrow$ $(\arg(z_1) \in 1.^\circ Q)$

$\Rightarrow \frac{\pi}{6}$ é um argumento de z_1

$z_1 = \text{cis}\left(\frac{\pi}{6}\right)$

Seja $z_2 = 1 + i$

$|z_2| = \sqrt{1^2 + 1^2} = \sqrt{2}$

$\begin{cases} \text{tg}(\arg z_2) = 1 \\ \arg z_2 \in 1.^\circ Q \end{cases} \Rightarrow \frac{\pi}{4}$ é um argumento de z_2

$z_2 = \sqrt{2}\,\text{cis}\left(\frac{\pi}{4}\right)$

17.3 $\left(\frac{-1+i^{15}}{\sqrt{3}+i}\right)^5 = \left(\frac{-1+\left(i^4\right)^3 \times i^3}{\sqrt{3}+i}\right)^5$

$= \left(\frac{-1+i}{\sqrt{3}+i}\right)^5 = \left[\frac{\sqrt{2}\text{cis}\left(\frac{5\pi}{4}\right)}{2\text{cis}\left(\frac{\pi}{6}\right)}\right]^5$

$= \left[\frac{\sqrt{2}}{2}\text{cis}\left(\frac{5\pi}{4} - \frac{\pi}{6}\right)\right]^5$

$= \left(\frac{\sqrt{2}}{2}\right)^5 \text{cis}\left(5\left(\frac{15\pi}{12} - \frac{2\pi}{12}\right)\right)$

$= \frac{4\sqrt{2}}{32}\text{cis}\left[5\left(\frac{13\pi}{12}\right)\right] = \frac{\sqrt{2}}{8}\text{cis}\left(\frac{65\pi}{12}\right)$

$= \frac{\sqrt{2}}{8}\text{cis}\left(\frac{17\pi}{12}\right)$

Cálculo auxiliar

Seja $z_1 = -1 - i$, então:

$|z_1| = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$

$\begin{cases} \text{tg}(\arg z_1) = \frac{-1}{-1} = 1 \\ \arg z_1 \in 3.^\circ Q \end{cases} \Rightarrow \pi + \frac{\pi}{4} = \frac{5\pi}{4}$ é um argumento de z_1

$z_1 = \text{cis}\left(\frac{\pi}{6}\right)$

Seja $z_2 = \sqrt{3} + i$, então:

$|z_2| = \sqrt{\left(\sqrt{3}\right)^2 + (+1)^2} = \sqrt{4} = 2$

$\begin{cases} \text{tg}(\arg z_2) = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} \\ \arg z_2 \in 1.^\circ Q \end{cases} \Rightarrow \frac{\pi}{6}$ é um argumento de z_2

logo $z_2 = 2\,\text{cis}\left(\frac{\pi}{6}\right)$

17.4 $\frac{i^{513} - (1+i)^4}{1+i} - i = \frac{(i)^4 \times i - (1+i)^4}{1+i}$

$= \frac{i - (1+i)^4}{1+i} - i = \frac{i+4}{1+i} - i$

$= \frac{i+4}{1+i} - \frac{i(1+i)}{(1+i)}$

$= \frac{i+4-i-i^2}{1+i} = -\frac{5}{1+i}$

$= \frac{5\text{cis}0}{\sqrt{2}\text{cis}\left(\frac{\pi}{4}\right)} = \frac{5}{\sqrt{2}}\text{cis}\left(-\frac{\pi}{4}\right)$

$= \frac{5\sqrt{2}}{2}\text{cis}\left(-\frac{\pi}{4}\right)$

Cálculo auxiliar

Seja $w = 1 + i$, então: $|w| = \sqrt{2}$.

$\begin{cases} \text{tg}(\arg w) = \frac{1}{1} = 1 \\ \arg w \in 1.^\circ Q \end{cases} \Rightarrow \frac{\pi}{4}$ é um argumento de w

$w = \sqrt{2}\,\text{cis}\left(\frac{\pi}{4}\right)$

$(1+i)^4 = w^4 = \left(\sqrt{2}\right)^4 \text{cis}\left(4 \times \frac{\pi}{4}\right) = 4\text{cis}\pi$

$= 4(\cos\pi + i\sin\pi) = 4(-1 + i \times 0) = -4$

17.5 $\left(\frac{3 - 4i - \frac{1}{i}}{-3i^{22}}\right)^5 = \left(\frac{3 - 4i - \frac{-i}{i(-i)}}{-3i^{22}}\right)^5$

$= \left(\frac{3 - 4i - \frac{-i}{-i^2}}{-3\left(i^4\right)^5 i^2}\right)^5 = \left(\frac{3 - 4i + i}{-3\left(i^2\right)}\right)^5$

$= \left(\frac{3-3i}{3}\right)^5 = (1-i)^5 = \left(\sqrt{2}\cos\left(-\frac{\pi}{4}\right)\right)^5$

$= 4\sqrt{2}\text{cis}\left(-\frac{5\pi}{4}\right) = 4\sqrt{2}\text{cis}\left(-\frac{5\pi}{4} + 2\pi\right)$

$= 4\sqrt{2}\,\text{cis}\left(\frac{3\pi}{4}\right)$

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18.1 a) $\sqrt{\sqrt{3}-\mathrm{i}}=\sqrt{2\operatorname{cis}\left(-\frac{\pi}{6}\right)}$

$=\sqrt{2}\operatorname{cis}\frac{-\frac{\pi}{6}+2k\pi}{2}, k\in\{0, 1\}$

Se $k=0$, $z_0=\sqrt{2}\operatorname{cis}\left(-\frac{\pi}{12}\right)$

Se $k=1$, $z_1=\sqrt{2}\operatorname{cis}\left(\frac{11\pi}{12}\right)$

b) $\sqrt[4]{\left(-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\mathrm{i}\right)}=\sqrt[4]{1\operatorname{cis}\left(\frac{3\pi}{4}\right)}$

$=\sqrt[4]{1}\operatorname{cis}\frac{\frac{3\pi}{4}+2k\pi}{4}, k\in\{0, 1, 2, 3\}$

$=\operatorname{cis}\frac{3\pi+8k\pi}{16}, k\in\{0, 1, 2, 3\}$

Se $k=0\rightarrow z_0=\operatorname{cis}\left(\frac{3\pi}{16}\right)$

Se $k=1\rightarrow z_1=\operatorname{cis}\left(\frac{11\pi}{16}\right)$

Se $k=2\rightarrow z_2=\operatorname{cis}\left(\frac{19\pi}{16}\right)$

Se $k=3\rightarrow z_3=\operatorname{cis}\left(\frac{27\pi}{16}\right)$

Cálculo auxiliar

Seja $w=\sqrt{3}-\mathrm{i}$, então:

$|w|=\sqrt{\left(\sqrt{3}\right)^2+(-1)^2}=2$

$\begin{cases}\operatorname{tg}(\arg w)=\frac{-1}{\sqrt{3}}=-\frac{\sqrt{3}}{3}\\ \arg w\in 4.^{\text{o}}Q\end{cases}\Rightarrow$

$\Rightarrow -\frac{\pi}{6}$ é um argumento de w

Seja $z=-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\mathrm{i}=-\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\mathrm{i}$

$|z|=\sqrt{\left(-\frac{\sqrt{2}}{2}\right)^2+\left(\frac{\sqrt{2}}{2}\right)^2}=\sqrt{\frac{2}{4}+\frac{2}{4}}=1$

$\begin{cases}\operatorname{tg}(\arg z)=\frac{\sqrt{2}}{-\sqrt{2}}=-1\\ \arg z\in 2.^{\text{o}}Q\end{cases}\Rightarrow$

$\Rightarrow \pi-\frac{\pi}{4}=\frac{3\pi}{4}$ é argumento de z

Logo $z=1\operatorname{cis}\left(\frac{3\pi}{4}\right)$

c) $\sqrt[5]{-32\mathrm{i}}=\sqrt[5]{32\operatorname{cis}\left(\frac{3\pi}{2}\right)}$

$=\sqrt[5]{32}\operatorname{cis}\frac{\frac{3\pi}{2}+2k\pi}{5}, k\in\{0, 1, 2, 3, 4\}$

$=2\operatorname{cis}\frac{\frac{3\pi}{2}+2k\pi}{5}, k\in\{0, 1, 2, 3, 4\}$

$=2\operatorname{cis}\frac{3\pi+4k\pi}{10}, k\in\{0, 1, 2, 3, 4\}$

Se $k=0\rightarrow z_0=2\operatorname{cis}\left(\frac{3\pi}{10}\right)$

Se $k=1\rightarrow z_1=2\operatorname{cis}\left(\frac{7\pi}{10}\right)$

Se $k=2\rightarrow z_2=2\operatorname{cis}\left(\frac{11\pi}{10}\right)$

Se $k=3\rightarrow z_3=2\operatorname{cis}\left(\frac{15\pi}{10}\right)$

Se $k=4\rightarrow z_4=2\operatorname{cis}\left(\frac{19\pi}{10}\right)$

18.2 a) $z^3-1=0\Leftrightarrow z^3=1\Leftrightarrow z=\sqrt[3]{1}$

$\Leftrightarrow z=\sqrt[3]{1\operatorname{cis}0}\Leftrightarrow z=\sqrt[3]{1}\operatorname{cis}\frac{0+2k\pi}{3}, k\in\{0, 1, 2\}$

$\Leftrightarrow z=\operatorname{cis}\left(\frac{2k\pi}{3}\right), k\in\{0, 1, 2\}$

$k=0\rightarrow z_0=\operatorname{cis}0$

$k=1\rightarrow z_1=\operatorname{cis}\frac{2\pi}{3}$

$k=2\rightarrow z_2=\operatorname{cis}\frac{4\pi}{3}$, logo

$S=\left\{\operatorname{cis}0, \operatorname{cis}\frac{2\pi}{3}, \operatorname{cis}\frac{4\pi}{3}\right\}$

b) $z^4+1=0\Leftrightarrow z^4=-1\Leftrightarrow z=\sqrt[4]{-1}$

$\Leftrightarrow z=\sqrt[4]{1\operatorname{cis}\pi}\Leftrightarrow$

$\Leftrightarrow z=\sqrt[4]{1}\operatorname{cis}\frac{\pi+2k\pi}{4}, k\in\{0,1,2,3\}$

$\Leftrightarrow z=\operatorname{cis}\frac{\pi+2k\pi}{4}, k\in\{0,1,2,3\}$

$k=0 \rightarrow z_0 = \text{cis}\,\frac{\pi}{4}$

$k=1 \rightarrow z_1 = \text{cis}\,\frac{3\pi}{4}$

$k=2 \rightarrow z_2 = \text{cis}\,\frac{5\pi}{4}$

$k=3 \rightarrow z_3 = \text{cis}\,\frac{7\pi}{4}$, logo

$S = \left\{\text{cis}\,\frac{\pi}{4}, \text{cis}\,\frac{3\pi}{4}, \text{cis}\,\frac{5\pi}{4}, \text{cis}\,\frac{7\pi}{4}\right\}$

c) $z^6 + \text{i} - 1 = 0 \Leftrightarrow z^6 = 1 - \text{i}$

$\Leftrightarrow z = \sqrt[6]{1-\text{i}}$

Como $1 - \text{i} = \sqrt{2}\text{cis}\left(-\frac{\pi}{4}\right)$, vem:

$z = \sqrt[6]{\sqrt{2}\text{cis}\left(-\frac{\pi}{4}\right)} \Leftrightarrow$

$\Leftrightarrow z = \sqrt[12]{2}\text{cis}\,\frac{-\frac{\pi}{4}+2k\pi}{6}, k \in \{0,1,2,3,4,5\}$

$\Leftrightarrow z = \sqrt[12]{2}\text{cis}\,\frac{-\pi+8k\pi}{24}, k \in \{0,1,2,3,4,5\}$

$k=0 \rightarrow z_0 = \sqrt[12]{2}\text{cis}\left(-\frac{\pi}{24}\right)$

$k=1 \rightarrow z_1 = \sqrt[12]{2}\text{cis}\left(\frac{7\pi}{24}\right)$

$k=2 \rightarrow z_2 = \sqrt[12]{2}\text{cis}\left(\frac{15\pi}{24}\right) = \sqrt[12]{2}\text{cis}\left(\frac{5\pi}{8}\right)$

$k=3 \rightarrow z_3 = \sqrt[12]{2}\text{cis}\left(\frac{23\pi}{24}\right)$

$k=4 \rightarrow z_4 = \sqrt[12]{2}\text{cis}\left(\frac{31\pi}{24}\right)$

$k=5 \rightarrow z_5 = \sqrt[12]{2}\text{cis}\left(\frac{39\pi}{24}\right) = \sqrt[12]{2}\text{cis}\left(\frac{13\pi}{8}\right)$

logo:

$$S = \left\{\sqrt[12]{2}\text{cis}\left(-\frac{\pi}{24}\right), \sqrt[12]{2}\text{cis}\left(\frac{7\pi}{12}\right), \sqrt[12]{2}\text{cis}\left(\frac{5\pi}{8}\right), \sqrt[12]{2}\text{cis}\left(\frac{23\pi}{24}\right), \sqrt[12]{2}\text{cis}\left(\frac{31\pi}{24}\right), \sqrt[12]{2}\text{cis}\left(\frac{13\pi}{8}\right)\right\}$$

d) $z^3 = \bar{\bar{z}} \Leftrightarrow z^3 = z$

$\Leftrightarrow z^3 - z = 0$

$\Leftrightarrow z\left(z^2 - 1\right) = 0$

$\Leftrightarrow z = 0 \vee z^2 = 1$

$\Leftrightarrow z = 0 \vee z = 1 \vee z = -1$

$S = \{-1, 0, 1\}$

e) $z = \bar{z}^2$

Fazendo $z = \rho\,\text{cis}\,\theta$, temos:

$z = \bar{z}^2 \Leftrightarrow \rho\,\text{cis}\,\theta = \left(\rho\,\text{cis}(-\theta)\right)^2 \Leftrightarrow$

$\Leftrightarrow \rho\,\text{cis}(\theta) = \rho^2\,\text{cis}(-2\theta) \Leftrightarrow$

$\Leftrightarrow \begin{cases} \rho = \rho^2 \\ \theta = -2\theta + 2k\pi,\ k \in \mathbb{Z} \end{cases} \Leftrightarrow \begin{cases} \rho^2 - \rho = 0 \\ 3\theta = 2k\pi,\ k \in \mathbb{Z} \end{cases} \Leftrightarrow$

$\Leftrightarrow \begin{cases} \rho(\rho - 1) = 0 \\ \theta = \frac{2k\pi}{3},\ k \in \mathbb{Z} \end{cases} \Leftrightarrow\Leftrightarrow \begin{cases} \rho = 0 \vee \rho = 1 \\ \theta = \frac{2k\pi}{3},\ k \in \mathbb{Z} \end{cases} \Leftrightarrow$

$\Leftrightarrow z = 0 \vee z = \text{cis}\,0 \vee z = \text{cis}\,\frac{2\pi}{3} \vee z = \text{cis}\,\frac{4\pi}{3}$

$S = \left\{0, \text{cis}\,0, \text{cis}\,\frac{2\pi}{3}, \text{cis}\,\frac{4\pi}{3}\right\}$

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19. $-64 = 64\ \text{cis}\ \pi$

$\sqrt[6]{-64} = \sqrt[6]{64\,\text{cis}\,\pi} = \sqrt[6]{64}\text{cis}\,\frac{\pi+2k\pi}{6} = 2\text{cis}\,\frac{\pi+2k\pi}{6}$,

$k \in \{0,\ 1,\ 2,\ 3,\ 4,\ 5\}$

Então:

$z_0 = 2\text{cis}\left(\frac{\pi}{6}\right)$

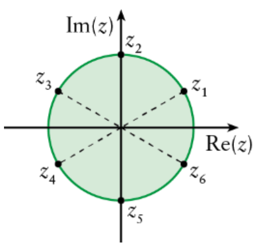
$z_1 = 2\text{cis}\left(\frac{3\pi}{6}\right) = 2\text{cis}\left(\frac{\pi}{2}\right)$

$z_2 = 2\text{cis}\left(\frac{5\pi}{6}\right)$

$z_3 = 2\text{cis}\left(\frac{7\pi}{6}\right)$

$z_4 = 2\text{cis}\left(\frac{9\pi}{6}\right) = 2\text{cis}\left(\frac{3\pi}{2}\right)$

$z_5 = 2\text{cis}\left(\frac{11\pi}{6}\right)$



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20.1 Sabemos que $z_1 = 1+\sqrt{3}\mathrm{i}$, daí que:

$$|z_1| = \sqrt{1^2 + \left(\sqrt{3}\right)^2} = 2$$

$$\mathrm{tg}\left(\arg\left(z_1\right)\right) = \frac{\sqrt{3}}{1} \Rightarrow \qquad \left(\arg(z_1) \in 1.^{\circ}Q\right)$$

$\Rightarrow \frac{\pi}{3}$ é um argumento de z_1

$$z_1 = 2\mathrm{cis}\left(\frac{\pi}{3}\right);$$

$$z_2 = 2\mathrm{cis}\left(\frac{\pi}{3}+\frac{2\pi}{3}\right) = 2\mathrm{cis}\left(\pi\right) = -2$$

$$z_3 = 2\mathrm{cis}\left(\pi+\frac{2\pi}{3}\right) = 2\mathrm{cis}\left(\frac{5\pi}{3}\right)$$

$$= 2\left(\frac{1}{2}-\frac{\sqrt{3}}{2}\mathrm{i}\right) = 1-\sqrt{3}\mathrm{i}$$

20.2. Calculando a soma, na forma algébrica $z_1 + w$, $z_2 + w$ e $z_3 + w$, obtém-se o transformado do triângulo de vértices z_1, z_2 e z_3 pela translação $T_{\vec{u}}$, assim:

$$z'_1 = z_1 + w = 1+\sqrt{3}\mathrm{i} + (-\mathrm{i}) = 1+\left(\sqrt{3}-1\right)\mathrm{i}$$

$$z'_2 = z_2 + w = -2 + (-\mathrm{i}) = -2 - \mathrm{i}\ \mathrm{i}$$

$$z'_3 = z_3 + w = 1-\sqrt{3}\mathrm{i} + (-\mathrm{i}) = 1+\left(-\sqrt{3}-1\right)\mathrm{i}$$

20.3. $z_1 \times z = 2\,\mathrm{cis}\left(\frac{\pi}{3}\right)\times \mathrm{cis}\left(\frac{\pi}{3}\right)$

$$= 2\,\mathrm{cis}\left(\frac{\pi}{3}+\frac{\pi}{3}\right)$$

$$= 2\,\mathrm{cis}\left(\frac{2\pi}{3}\right)$$

$$z_2 \times z = 2\,\mathrm{cis}\left(\pi\right)\times \mathrm{cis}\left(\frac{\pi}{3}\right)$$

$$= 2\,\mathrm{cis}\left(\pi+\frac{\pi}{3}\right)$$

$$= 2\,\mathrm{cis}\left(\frac{4\pi}{3}\right)$$

$$z_3 \times z = 2\,\mathrm{cis}\left(\frac{5\pi}{3}\right)\times \mathrm{cis}\left(\frac{\pi}{3}\right) = 2\,\mathrm{cis}\left(\frac{5\pi}{3}+\frac{\pi}{3}\right)$$

$$= 2\,\mathrm{cis}\left(2\pi\right)$$

Interpretação: O produto de $\mathrm{cis}\left(\frac{\pi}{3}\right)$ por cada número complexo z cuja imagem, no plano complexo, é um vértice do triângulo, é o número complexo cuja imagem é o transformado de z na rotação de centro (0, 0) e ângulo igual a $\frac{\pi}{3}$.

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1. Seja $z = r\mathrm{cis}\theta$,então o seu simétrico é:

$$-z = r\mathrm{cis}\left(\pi+\theta\right)$$

Logo, a resposta correcta é a (B), uma vez que:

$$\mathrm{cis}\left(\alpha-\pi\right) = \mathrm{cis}\left(\alpha+2\pi-\pi\right) = \mathrm{cis}\left(\alpha+\pi\right)$$

Resposta: (B).

2. $z = 3\mathrm{cis}\left(\theta-\frac{\pi}{3}\right)$

Para que z seja imaginário puro de coeficiente positivo, temos:

$$\theta-\frac{\pi}{3} = \frac{\pi}{2} + 2k\pi, k\in\mathbb{Z}$$

$$\Leftrightarrow \theta = \frac{5\pi}{6} + 2k\pi, k\in\mathbb{Z}$$

Por exemplo, $\theta = \frac{5\pi}{6}$

Resposta: (B).

3. $z = \mathrm{i}^{20} + \mathrm{i}$

$z = (\mathrm{i}^4)^5 + \mathrm{i}$

$z = 1 + \mathrm{i}$

módulo: $|z| = \sqrt{2}$

argumento: $\arg z = \frac{\pi}{4}$:

$$z = \sqrt{2}\mathrm{cis}\left(\frac{\pi}{4}\right)$$

Resposta: (C).

4. As imagens geométricas de números conjugados são simétricas relativamente ao eixo real.

Resposta: (D).

5. $2w\mathrm{cis}\left(-\pi\right) = -2w$, é dobro do simétrico de w.

Resposta: (C).

6. $\mathrm{i} - w = \mathrm{i} - (2 + \mathrm{i}) = \mathrm{i} - 2 - \mathrm{i} = -2$

Donde $\mathrm{i} - w$ é um número real.

Resposta: (A).

7. Sendo $z = 2\text{cis}\left(\frac{\pi}{9}\right)$,

então, $\frac{1}{z^2} = \frac{1}{4\text{cis}\left(\frac{2\pi}{9}\right)} = \frac{1\text{cis}0}{4\text{cis}\left(\frac{2\pi}{9}\right)}$

$$= \frac{1}{4}\text{cis}\left(-\frac{2\pi}{9}\right)$$

Resposta: (C).

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1.1 $(z-3)^2(1-2\mathrm{i})^2 - \mathrm{i}^9$

$= [(3-\mathrm{i})-3]^2(1-2\mathrm{i})^2 - (\mathrm{i}^4)^2 \times \mathrm{i}$

$= (-\mathrm{i})^2(1-4\mathrm{i}+4\mathrm{i}^2) - \mathrm{i}$

$= -(1-4\mathrm{i}-4) - \mathrm{i}$

$= 3 + 4\mathrm{i} - \mathrm{i}$

$= 3 + 3\mathrm{i}$

1.2 Inverso do conjugado de w é $\frac{1}{\overline{w}}$, isto é, $\frac{1}{3-3\mathrm{i}}$, donde:

$$\frac{1}{3-3\mathrm{i}} = \frac{3+3\mathrm{i}}{(3-3\mathrm{i})(3+3\mathrm{i})} = \frac{3+3\mathrm{i}}{9+(3\mathrm{i})^2}$$

$$= \frac{3+3\mathrm{i}}{9-9\mathrm{i}^2} = \frac{3+3\mathrm{i}}{18} = \frac{3}{18} + \frac{3}{18}\mathrm{i} = \frac{1}{6} + \frac{1}{6}\mathrm{i}$$

Na forma trigonométrica:

$$\left|\frac{1}{\overline{w}}\right| = \sqrt{\left(\frac{1}{6}\right)^2 + \left(\frac{1}{6}\right)^2} = \sqrt{\frac{1}{36}+\frac{1}{36}} = \frac{\sqrt{2}}{6}$$

$$\begin{cases} \text{tg}\left(\arg\left(\frac{1}{\overline{w}}\right)\right) = 1 \\ \arg\left(\frac{1}{\overline{w}}\right) \in 1.^\circ Q \end{cases} \Rightarrow \frac{\pi}{4} \text{ é um argumento de } \frac{1}{\overline{w}}$$

Daí que: $\frac{1}{\overline{w}} = \frac{\sqrt{2}}{6}\text{cis}\left(\frac{\pi}{4}\right)$.

2.1 O polinómio $x^3 - 3x^2 + x + 5$ é divisível por $x + 1$, porque $P(-1) = 0$.

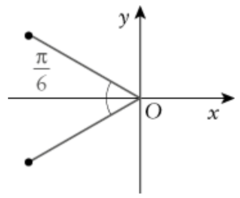
2.2 $x^3 - 3x^2 + x + 5 = (x+1)(x^2 - 4x + 5)$

$$x^2 - 4x + 5 = 0 \Leftrightarrow x = \frac{4 \pm \sqrt{16-20}}{2} \Leftrightarrow$$

$$\Leftrightarrow x = 2 + \mathrm{i} \vee x = 2 - \mathrm{i}$$

Os zeros do polinómio são -1, $2 + \mathrm{i}$ e $2 - \mathrm{i}$.

3. Do enunciado podemos retirar que:

- $|z| = |\overline{z}| = 1$
- $z = \text{cis}\,\theta$ com $\frac{\pi}{2} < \theta < \pi$.
- 

As imagens de z e $\overline{z}$ são simétricas em relação ao eixo real. Donde: $\arg(z) = \pi - \frac{\pi}{6} = \frac{5\pi}{6} \Rightarrow \arg(\overline{z}) = -\frac{5\pi}{6}$

3.1 $\mathrm{i}z = \text{cis}\frac{\pi}{2}\text{cis}\left(\frac{5\pi}{6}\right) = \text{cis}\left(\frac{\pi}{2} + \frac{5\pi}{6}\right)$

$$= \text{cis}\left(\frac{4\pi}{3}\right) = -\frac{1}{2} - \frac{\sqrt{3}}{2}\mathrm{i}$$

3.2 $\frac{\overline{z}}{\mathrm{i}} = \frac{\text{cis}\left(-\frac{5\pi}{6}\right)}{\text{cis}\left(\frac{\pi}{2}\right)} = \text{cis}\left(-\frac{4\pi}{6} - \frac{\pi}{2}\right)$

$$\text{cis}\left(-\frac{4\pi}{3}\right) = -\frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}$$

4. $z_1^3 = z_2^2 \times w$

$$\Leftrightarrow (2+\mathrm{i}) = \left(\sqrt{3}\text{cis}\left(\frac{\pi}{3}\right)\right)^2 w$$

$$\Leftrightarrow (2+\mathrm{i})^2(2+\mathrm{i}) = 3\text{cis}\frac{2\pi}{3}w$$

$$\Leftrightarrow (4+4\mathrm{i}+\mathrm{i}^2)(2+\mathrm{i}) = 3\text{cis}\frac{2\pi}{3}w$$

$$\Leftrightarrow (3+4\mathrm{i})(2+\mathrm{i}) = 3\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}\mathrm{i}\right)w$$

$$\Leftrightarrow (6+3\mathrm{i}+8\mathrm{i}+4\mathrm{i}^2) = -\frac{3}{2}+\frac{3\sqrt{3}}{2}\mathrm{i}$$

$$\Leftrightarrow (2+11\mathrm{i}) = -\frac{3}{2}+\frac{3\sqrt{3}}{2}\mathrm{i}$$

$$\Leftrightarrow \frac{2+11\mathrm{i}}{-\frac{3}{2}+\frac{3\sqrt{3}}{2}\mathrm{i}} = w$$

$$\Leftrightarrow \frac{(2+11i)\left(-\frac{3}{2}+\frac{3\sqrt{3}}{2}i\right)}{\frac{9}{4}+\frac{27}{4}}$$

$$\Leftrightarrow w = \frac{-3-3\sqrt{3}i-\frac{32}{2}i+\frac{33\sqrt{3}}{2}}{9}$$

$$\Leftrightarrow w = \frac{11\sqrt{3}-2}{6}-\frac{2\sqrt{3}+11}{6}i$$

5. $z_1 = 3i$

Portanto $z_2 = -3i$ é também um número complexo cuja imagem geométrica é um vértice do losango.

O perímetro do losango é 16. Logo, donde $\overline{AB} = 4$.

$$\overline{AB}^2 = \overline{OB}^2 + \overline{OA}$$

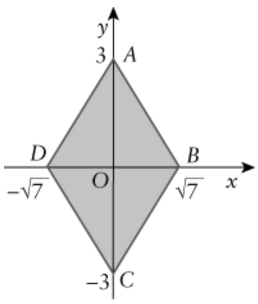
$$16 = 9 + \overline{OB}^2 \Leftrightarrow$$

$$\Leftrightarrow 16-9 = \overline{OB}^2 \Leftrightarrow$$

$$\Leftrightarrow 7 = \overline{OB}^2 \Leftrightarrow$$

Portanto, $\overline{OB} = \sqrt{7}$

Resposta: $-3i$, $\sqrt{7}$ e $-\sqrt{7}$



6. $$\left(\sqrt{2}\text{cis}\frac{\pi}{4}\right)^3 z = 1+i$$

$$\Leftrightarrow \left[\left(\sqrt{2}\right)^3 \text{cis}\left(\frac{3\pi}{4}\right)\right] z = \sqrt{2}\text{cis}\frac{\pi}{4}$$

$$\Leftrightarrow z = \frac{\sqrt{2}\text{cis}\frac{\pi}{4}}{\left(\sqrt{2}\right)^3 \text{cis}\frac{3\pi}{4}}$$

$$\Leftrightarrow z = \frac{\text{cis}\frac{\pi}{4}}{2\text{cis}\frac{3\pi}{4}} \Leftrightarrow z = \frac{1}{2}\text{cis}\left(-\frac{\pi}{2}\right)$$

7. Sabemos que:

- $z_1 = \overline{OA}\,\text{cis}\left(\frac{\pi}{4}\right)$
- $z_2 = 6\sqrt{2}\text{cis}(\theta_2)$
- a área do rectângulo $[OABC]$ é 12.

Assim, vem:

$$A_{[OABC]} = \overline{OA}\times\overline{OC} \Leftrightarrow$$

$$\Leftrightarrow 12 = \overline{OA}\times 6\sqrt{2} \Leftrightarrow \overline{OA} = \frac{12}{6\sqrt{2}}$$

$$\Leftrightarrow \overline{OA} = \frac{2}{\sqrt{2}} \Leftrightarrow \overline{OA} = \frac{2\sqrt{2}}{2} \Leftrightarrow \overline{OA} = \sqrt{2}$$

Assim: $z_1 = \sqrt{2}\text{cis}\left(\frac{\pi}{4}\right) = \sqrt{2}\left[\cos\left(\frac{\pi}{4}\right)+i\sin\left(\frac{\pi}{4}\right)\right]$

$$= \sqrt{2}\left(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right) = 1+i$$

Como $\arg(z_1) = \frac{\pi}{4}$ e $A\hat{O}C = \frac{\pi}{2}$, então:

$\arg(z_2) = \frac{\pi}{4}+\frac{\pi}{2}$, logo $\arg(z_2) = \frac{3\pi}{4}$

$$z_2 = 6\sqrt{2}\text{cis}\left(\frac{3\pi}{4}\right) = 6\sqrt{2}\left[\cos\left(\frac{3\pi}{4}\right)+i\sin\left(\frac{3\pi}{4}\right)\right]$$

$$= 6\sqrt{2}\left(-\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right) = -6+6i$$

Resposta: $z_1 = 1 + i$ e $z_2 = -6 + 6i$

8. z_1 tem um argumento $\frac{\pi}{3}$, logo $z_1 = r_1\text{cis}\frac{\pi}{3}$

z_2 tem módulo $3\sqrt{3}$, logo $z_2 = 3\sqrt{3}\text{cis}\theta_2$

Então:

$$z_2 \times \overline{z}_2 + \left(\frac{z_1}{|z_1|}\right)^{10} =$$

$$= 3\sqrt{3}\text{cis}(\theta_2)\times 3\sqrt{3}\,\text{cis}(-\theta_2) + \left(\frac{r_1\text{cis}\frac{\pi}{3}}{r_1}\right)^{10}$$

$$= \left(3\sqrt{3}\right)^2 \text{cis}(0) + \left(\text{cis}\frac{\pi}{3}\right)^{10}$$

$$= 27\text{cis}(0) + 1^{10}\,\text{cis}\frac{10\pi}{3}$$

$$= 27(\cos 0 + i\sin 0) + \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}$$

$$= 27(1 + i\times 0) + \left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)$$

$$= 27 - \frac{1}{2} - \frac{\sqrt{3}}{2}i = \frac{53}{2} - \frac{\sqrt{3}}{2}i$$

9. A imagem geométrica de *z* pertence ao 4.° quadrante (eixos não incluídos). Logo, *z* é da forma:

$z = r\text{cis}\theta$, com $-\frac{\pi}{2} < \theta < 0$

$z^3 = r^3\text{cis}(3\theta)$

E como $-\frac{\pi}{2} < \theta < 0$, então: $-\frac{3\pi}{2} < 3\theta < 0$.

Daí que a imagem geométrica de $z^3 = r^3\text{cis}(3\theta)$,

com $-\frac{3\pi}{2} < 3\theta < 0$, não pode pertencer ao 1.°

quadrante, mas pode pertencer ao 2.°, 3.° ou 4.° quadrantes.

10. Seja $z = r\text{cis}\theta$, tal que, $\theta = \frac{3\pi}{4}$ ou $\theta = -\frac{\pi}{4}$, uma vez que a imagem geométrica de *z* pertence à bissetriz dos quadrantes pares.

Assim:

$z^{20} = r^{20}\text{cis}\left(\frac{60\pi}{4}\right)$ ou $z^{20} = r^{20}\text{cis}\left(-\frac{20\pi}{4}\right)$

$z^{20} = r^{20}\text{cis}(15\pi)$ ou $z^{20} = r^{20}\text{cis}(-5\pi)$

ou seja:

$z^{20} = r^{20}\text{cis}(\pi)$ ou $z = r^{20}\text{cis}(-\pi)$

Logo, a imagem geométrica de z^{20} pertence ao eixo real, uma vez que $\arg(z^{20}) = \pi$ ou $\arg(z^{20}) = -\pi$

Então, uma equação da recta à qual pertence a imagem geométrica de z^{20} é $y = 0$ ou $\text{Im}(z) = 0$.